attention of college men fixed upon the varied and special problems of college work in mathematics, and with many group organizations for interchange of ideas, a new era will be inaugurated in this field.

3. **The Stimulation of National Organization.** The most natural climax of wide group organization is a national organization, such as was referred to in the October issue, and conversely, such a national organization will react on the formation of smaller groups and both will provide a far-reaching stimulus to individual activity. The MONTHLY would have welcomed the incorporation of such an organization within the American Mathematical Society, but since this is not to be, we look forward with high hopes and great enthusiasm to the organization of a new national society, and with more than four hundred charter members, we see no reason why commendable things should not be accomplished through this movement for the cause of mathematics in America.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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X.

E. POST-CANTORION DISSENSIONS (Concluded).

With the advent of the new century, discussion on Zeno began to quiet down in France. We note only two articles. In 1907 O. Hamelin wrote on the "Arrow," but the interest of his article centers in what constitutes the most probable renderings of the Aristotelian text.[1] In 1909 a novel attempt to solve Zeno's puzzles was made by Dunan[2] in an article in which he retracts what he said on this subject in a pamphlet of 1884.[3]

He believes that the difficulties vanish, on the recognition that motion takes place through a space, one and indivisible, without succession and parts. He admits that such a proposition raises considerable difficulty, which cannot be removed except by long and elaborate metaphysics, of which he gives in his article only a bare sketch.

No less radical is the position of Henri Bergson. He holds that philosophy must get back to reality itself. Reality is supplied by intuition. Pure intuition, external or internal, is that of undivided continuity. Every movement, in as much as it is a passage from rest to rest, is in fact absolutely indivisible. Sight perceives the movement in the form of a line which is traversed, and this line, like all space, may be indefinitely divided. We must not confound the data of

[1] *L'année philosophique* de F. Pillon, Paris, 1907, pp. 39–44.

[2] "Zénon d'Élée et le Nativism" in *Annales de Philosophie Chrétienne*, 1909.

[3] *Les arguments de Zénon d'Élée contre le mouvement*, Nantes, 1884.

the senses, which perceive the movement as an undivided whole, with the artifice of the mind which divides into parts the path traversed. Says Bergson:[1]

"You substitute the path for the journey, and because the journey is subtended by the path you think that the two coincide. But how should a *progress* coincide with a *thing*, a movement with an immobility? . . . And from the fact that this line is divisible into parts and that it ends in points, we cannot conclude either that the corresponding duration is composed of separate parts or that it is limited by instants. The arguments of Zeno of Elea have no other origin than this illusion. They all consist in making time and movement coincide with the line which underlies them, in attributing to them the same subdivisions as to the line, in short in treating them like that line. In this confusion Zeno was encouraged by common sense, which usually carries over to the movement the properties of its trajectory, and also by language, which always translates movement and duration in terms of space. . . . But the philosopher who reasons upon the inner nature of movement is bound to restore to it the mobility which is its essence, and this is what Zeno omits to do. By the first argument (the Dichotomy) he supposes the moving body to be at rest, and then considers nothing but the stages, infinite in number, that are along the line to be traversed: we cannot imagine, he says, how the body could ever get through the interval between them. But in this way he merely proves that it is impossible to construct, *à priori*, movement with immobilities, a thing no man ever doubted. The sole question is whether, movement being posited as a fact, there is a sort of retrospective absurdity in assuming that an infinite number of points has been passed through. But at this we need not wonder, since movement is an undivided fact, or a series of undivided facts, whereas the trajectory is infinitely divisible. In the second argument (the Achilles) movement is indeed given, it is even attributed to two moving bodies, but, always by the same error, there is an assumption that their movement coincides with their path, and that we may divide it, like the path itself, in any way we please. Then, instead of recognizing that the tortoise has the pace of a tortoise and Achilles the pace of Achilles, so that after a certain number of these indivisible acts or bounds Achilles will have outrun the tortoise, the contention is that we may disarticulate as we will the movement of Achilles and, as we will also, the movement of the tortoise: thus reconstructing both in an arbitrary way, according to a law of our own which may be incompatible with the real conditions of mobility. The same fallacy appears, yet more evident, in the third argument (the Arrow) which consists in the conclusion that, because it is possible to distinguish points on the path of a moving body, we have the right to distinguish indivisible moments in the duration of its movement. But the most instructive of Zeno's arguments is perhaps the fourth (the Stadium) which has, we believe, been unjustly disdained, and of which the absurdity is more manifest only because the postulate masked in the three others is here frankly displayed. Without entering on a discussion which would here be out of place, we will content ourselves with observing that motion, as given to spontaneous perception, is a fact which is quite clear, and that the difficulties and contradictions pointed out by the Eleatic school concern far less the living movement itself than a dead and artificial reorganization of movement by the mind."

Bergson discusses the "Arrow" more fully in his *L'Evolution creatrive*, 1907, where he refers[2] to the absurdity of regarding movement as made up of immobilities. He says:

"Philosophy perceived this as soon as it opened its eyes. The arguments of Zeno of Elea although formulated with a different intention, have no other meaning. . . . Motionless in each point of its course, it is motionless during all the time of its moving. Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course. The most that we can say is that it might be there, in this sense, that it passes there and might stop there. . . . You fix a point *C* in the interval passed, and say that at a certain moment the arrow was at *C*. If it had been there it would have been stopped there, and you would no longer have had a flight from *A* to *B*, but *two* flights, one from *A* to *C* and the other from *C* to *B*, with an interval of rest. A single movement is entirely, by the hypothesis, a movement between two stops; if there are intermediate stops, it is no longer a single movement."

[1] H. Bergson, *Matter and Memory*, transl. by Nancy M. Paul and W. Scott Palmer, London, 1911, pp. 248, 250–253. The first French edition appeared in 1896.

[2] H. Bergson, *Creative Evolution*, transl. by A. Mitchell, London, 1911, pp. 325–327.

There have been many dissussions of Bergson. One writer endeavors to point out his errors by returning to the continuums of Artistotle and Thomas Aquinas.[1] Most pertinent to our topic are the criticisms by Bertrand Russell, of Cambridge, England, which are displayed by the following quotations:[2]

". . . it will be said, the arrow is where it is at any one moment, but at another moment it is somewhere else, and this is just what constitutes motion. Certain difficulties, it is true, arise out of the continuity of motion, if we insist upon assuming that motion is also discontinuous. These difficulties, thus obtained, have long been part of the stock-in-trade of philosophers. But if, with the mathematicians, we avoid the assumption that motion is also discontinuous, we shall not fall into the philosopher's difficulties. A cinematograph in which there are an infinite number of films, and in which there is never a *next* film because an infinite number come between any two, will perfectly represent a continuous motion. Wherein, then, lies the force of Zeno's argument? . . . Zeno assumes, tacitly, the essence of the Bergsonian theory of change. That is to say, he assumes that when a thing is in process of continuous change, even if it is only change of position, there must be in the thing some internal *state* of change. The thing must, at each instant, be intrinsically different from what it would be if it were not changing. He then points out that at each instant the arrow simply is where it is, just as it would be if it were at rest. Hence he concludes that there can be no such thing as a *state* of motion, and therefore, adhering to the view that a state of motion is essential to motion, he infers that there can be no motion and that the arrow is always at rest. Zeno's argument, therefore, though it does not touch the mathematical account of change, does, *prima facie*, refute a view of change which is not unlike M. Bergson's. How, then, does M. Bergson meet Zeno's argument? He meets it by denying that the arrow is ever anywhere. After stating Zeno's argument, he replies: 'Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course.' (C. E., p. 325.) This reply to Zeno, or a closely similar one concerning Achilles and the Tortoise, occurs in all his three books. Bergson's view plainly, is paradoxical; whether it be *possible*, is a question which demands a discussion of his view of duration. His only argument in its favor is the statement that the mathematical view of change 'implies the absurd proposition that movement is made of immobilities.' (C. E., p. 325.) But the apparent absurdity of this view is merely due to the verbal form in which he has stated it, and vanishes as soon as we realize that motion implies relations. A friendship, for example, is made out of people who are friends, but not out of friendships. . . . So a motion is made out of what is moving, but not out of motions. It expresses the fact that a thing may be in different places at different times, and that the places may still be different however near together the times may be. Bergson's argument against the mathematical view of motion, therefore, reduces itself, in the last analysis, to a mere play upon words." . . .

"Mathematics conceives change, even continuous change, as constituted by a series of states; Bergson, on the contrary, contends that no series of states can represent what is continuous, and that in change a thing is never in any state at all." . . .

"One of the bad effects of an anti-intellectual philosophy, such as that of Bergson, is that it thrives upon the errors and confusions of the intellect. Hence it is led to prefer bad thinking to good, to declare every momentary difficulty insoluble, and to regard every foolish mistake as revealing the bankruptcy of intellect and the triumph of intuition. . . . As regards mathematics, he has deliberately preferred traditional errors in interpretation to the more modern views which have prevailed among mathematicians for the last half century."

Thus it is seen that among recent French philosophers the Cantor continuum has been neglected and no satisfactory substitute has been advanced.

A treatment of the "Achilles" altogether different from that hitherto given by either philosophers or mathematicians is given by Russell.[3] After explaining infinite number and the modern continuum, he says in the *International Monthly:*

[1] T. J. Gerrard, *Bergson, an Exposition and Criticism*, London and Edinburgh, 1913, pp. 23 ff.

[2] B. Russell, "The Philosophy of Bergson," *The Monist*, July, 1912. Russell's references (C. E.) are to the English translations of Bergson's *Creative Evolution.*

[3] See B. Russell, *Principles of Mathematics*, 1902; "Recent work on the Principles of Mathematics" in the *International Monthly*, Vol. IV, 1901, pp. 83–101.

"We can now understand why Zeno believed that Achilles cannot overtake the tortoise and why as a matter of fact he can overtake it. We shall see that all the people who disagreed with Zeno had no right to do so, because they all accepted premises from which his conclusion followed. . . . Then he [Achilles] will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another place in another moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise should have been, would be only part of the places, where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion. The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities."

The conjectures which Russell makes on the history of the "Achilles" are, in the main, without foundation. There is no historical evidence for believing that Zeno based the "Achilles" on the doctrine that the whole is greater than any of its parts. Aristotle bases Zeno's argument on the assertion that a line or distance cannot be reduced by any process of successive division to elements that are mathematical points. Russell's version of the paradox is what Zeno might have said, but did not actually say. It is far simpler to explain than is that of Zeno. Assent can be readily secured to the fact that, in infinite aggregates, the whole is not greater than certain of its parts.[1]

That Russell's argument, though correct in itself, does not meet the exact difficulty experienced by many persons, is brought out by C. D. Broad,[2] who points out that the "Achilles" rests on the false assumption that "what is beyond every one of an infinite series of points, must be infinitely beyond the first point of the series." Broad considers it important even at this time to settle this controversy, "because it and Zeno's other paradoxes have become the happy hunting-ground of Bergsonians and like contemners of the human intellect." What makes infinite divisibility a stumbling block to so many is the fact that appeal is made to sensory intuition and imagination—the very faculty of the mind which proves itself unable to cope with the problem. But our powers of analysis penetrate realms of thought beyond the reach of the imagination, and it is in that territory that the arguments of Zeno are made to surrender their mysteries.

B. Russell took great interest also in the "Arrow." In the *International Review* he remarked:

[1] In this connection a story told by De Morgan may be of interest. He relates "a tradition of a Cambridge professor who was once asked in a mathematical discussion, 'I suppose you will admit that the whole is greater than its part?' and who answered, "not I, until I see what use you are going to make of it.' " The danger of unintended implications is illustrated by an author who remarked that Gibbon always had a copy of Horace in his pocket and often in his hand, from which it would seem to follow that Gibbon's hand was sometimes in his pocket.

[2] *Mind*, Vol. 38, 1913, pp. 318, 319.

"Weierstrass, by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world, and that the arrow in its flight is truly at rest. Zeno's only error lay in inferring (if he did infer) that, because there is no change, therefore the world is in the same state at any one time as at any other. . . . Weierstrass has been able, by embodying his views in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, to invest Zeno's paradoxes with the respectable air of platitudes." Elsewhere Russell expresses much the same idea by the statement that "a variable does not vary."[1]

That a variable cannot reach its limit is still widely held. In 1907 R. B. Haldane presented this as the teaching of mathematics, in a presidential address to the Aristotelian Society, entitled "The Methods of Modern Logic and the Conception of Infinity." In a review of this address, B. Russell says[2] that this property "belongs to limits of a certain particular sort," which constitute "an extremely special case, not realized in most of the series in which limits exist."

The creation of the theory of sets and of the Cantor continuum lead to modified definitions of the limit. In this theory the concept of a limit was divorced from the idea of quantity and measurement. The question whether the variable reaches its limit or not is ignored as being of no interest. Whether it reaches its limit or not depends upon the nature of the variation in a particular case; the sequence of values may include the limit, or it may not. The *limiting point* of a set of points is one for which every interval, however small, containing the limiting point, encloses a point of the set, distinct from the limiting point itself. A *limit* is merely the arithmetical equivalent of the limiting point in geometry. The introduction of a transfinite number as a limit has carried with it still further modification of the idea of a limit. Small intervals do not fit here. Says Bertrand Russell: "If we consider the whole series of integers, finite and infinite, arranged in order of magnitude, then the class of finite integers, considered as part of this series, has an upper limit, namely the smallest of the infinite integers (which is the number of finite integers)." Here there is no "negligible difference" between variable and limit; "the difference between the finite integers and their limit remains constant and infinite." Again he says: "A limit must not be conceived as something to which the successive terms of the class approach indefinitely near; they may all be at an infinite distance from the limit, or at a distance which remains permanently greater than some given finite distance; or the series concerned may be one in which there is no such thing as distance or difference." His definition of a limit is as follows: "Given any series, and a class α of terms belonging to the series, a term x belonging to the series is called the *upper limit* of α if every term of α precedes x, and every term of the series which precedes x precedes some member of α." He gives a similar definition for *lower limit*.[3] It is to be observed that the modern definitions of a limit are free from the concept of the old-time infinitesimal.

As now we pause and look backward, we see that a full and logically correct explanation of Zeno's arguments on motion has been given by the philosophers of mathematics. Looking about us, we see that the question is still regarded as

[1] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, p. 240.
[2] *Mind*, Vol. 33, 1908, p. 239.
[3] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, pp. 240, 241.

being in an unsettled condition. Philosophers whose intellectual interests are remote from mathematics are taking little interest in the linear continuum as created by the school of Georg Cantor. Nor do they offer a satisfactory substitute. The main difficulty is not primarily one of logic; it is one of postulates or assumptions. What assumptions are reasonable and useful? On this point there is disagreement. Cantor and his followers are willing to assume a continuum which transcends sensuous intuition. Others are not willing to do so. Hence the divergence. In the Koran there is a story that, after the creation of Adam, the angels were commanded to make him due reverence. But the chief of the angels refused, saying: "Far be it from me a pure spirit to worship a creature of clay." For this refusal he was shut out from Paradise. The doom of that chief, so far as the mathematical paradise is concerned, awaits those who refuse to examine with proper care the massive creation by our great mathematicians, without which the tiniest quiver of a leaf on a tree remains incomprehensible.[1]

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MATHEMATICAL MEETINGS IN CALIFORNIA.

I. THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society met for its twenty-second summer meeting as announced by the Society, on August 3, 1915, at the University of California in Berkeley. The first meeting was in conjunction with Section A of the American Association for the Advancement of Science, on Tuesday morning. Professor Keyser, of Columbia University, delivered an address on the human significance of mathematics, and Director Hale, of the Mount Wilson Solar Observatory, delivered an address on the work of a modern observatory. The attendance was very large and included members of the American Mathematical

[1] Since the completion of this article there has appeared Bertrand Russell's *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*, Open Court Company, 1914, in which much attention is given to Zeno's arguments. An article on Zeno by Philip E. B. Jourdain will soon appear in *Mind*.